

$$\begin{aligned} \frac{dK(k)}{dk} &= \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k} = \frac{\pi}{4} k \left(1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 \right. \\ &\quad \left. + \frac{19845}{16384} k^8 + \frac{160083}{131072} k^{10} + \frac{1288287}{1048576} k^{12} + \frac{41409225}{33554432} k^{14} + \dots \right) \\ &= \frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \right]^2 n k^{2n-1} = \frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^{n-1} n!} \right]^2 n k^{2n-1}; \\ \frac{dE(k)}{dk} &= \frac{E(k) - K(k)}{k} = -\frac{\pi}{4} k \left(1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{175}{1024} k^6 + \dots \right) \\ &= -\frac{2205}{16384} k^8 - \frac{14553}{131072} k^{10} - \frac{99099}{1048576} k^{12} - \frac{2760615}{33554432} k^{14} + \dots \\ &= -\frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \right]^2 \frac{n k^{2n-1}}{2n-1} = -\frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^{n-1} n!} \right]^2 \frac{n k^{2n-1}}{2n-1}. \end{aligned}$$

At $k=0$: $dK/dk = dE/dk = 0$; at $k=1$: $dK/dk \uparrow \infty$; $dE/dk \downarrow (-\infty)$. Applying the previous two exact relations and using the four definitions from chapter 2 one gets the expansions:

$$\begin{aligned} \left[\frac{dK(k)}{dk} \right]_0 &= \frac{\pi}{4} k \left(1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225 \cdot 75}{1024} k^6 + \dots \right); \\ \left[\frac{dE(k)}{dk} \right]_0 &= -\frac{\pi}{4} k \left(1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{174 \cdot 25}{1024} k^6 + \dots \right), \end{aligned}$$

for the 1st set of approximate functions (K_0, E_0), and resp.

$$\begin{aligned} \left[\frac{dK(k)}{dk} \right]_1 &= \frac{\pi}{4} k \left(1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 + \frac{19845}{16384} k^8 \right. \\ &\quad \left. + \frac{160083}{131072} k^{10} + \frac{1288287}{1048576} k^{12} + \frac{41409226125}{33554432} k^{14} + \dots \right); \\ \left[\frac{dE(k)}{dk} \right]_1 &= -\frac{\pi}{4} k \left(1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{175}{1024} k^6 + \frac{2205}{16384} k^8 \right. \\ &\quad \left. + \frac{14553}{131072} k^{10} + \frac{99099}{1048576} k^{12} + \frac{276061425}{33554432} k^{14} + \dots \right), \end{aligned}$$

for the 2nd set of approximate functions (K_1, E_1). The difference with respect to the expansions of the 1st derivatives of the exact functions (K, E) begins at the terms in k^7 for the 1st set, and at the terms in k^{15} for the 2nd one, so much lesser than that for the expansions of the respective sets ($K_{0,1}, E_{0,1}$). One can also easily find the analytic expressions and series representations for the 2nd derivatives of all $K, K_{0,1}, E, E_{0,1}$, with similar results, but a lesser precision than for K, E, K', E' . Besides the above definitions of the derivatives $K' (= dK/dk)$, $E' (= dE/dk)$, there is a useful functional relation (Legendre's): $K(k) \cdot E(k') + E(k) \cdot K(k') - K(k) \cdot K(k') = \pi/2$.

V. GRAPHIC COMPARISON

The variation curves of Legendre complete elliptic integrals, as well as that of the two sets of closed analytic functions are graphically represented in the comparative figures 1 and 2, all vs. θ , in sexagesimal degrees, and given by $\theta = \sin^{-1} k$. In both figures the exact functions $K(k), E(k)$ were represented by solid (continuous) black lines, the 1st set of approximation [$K_0(k), E_0(k)$] by dashed black lines, and the 2nd set of approximation [$K_1(k), E_1(k)$] by solid red lines. At $k=1$ the graphs of all $K_{0,1}(k)$ fall to $(-\infty)$; the graphs of all $E_{0,1}(k)$ pass through $(1, 3\pi/8)$.

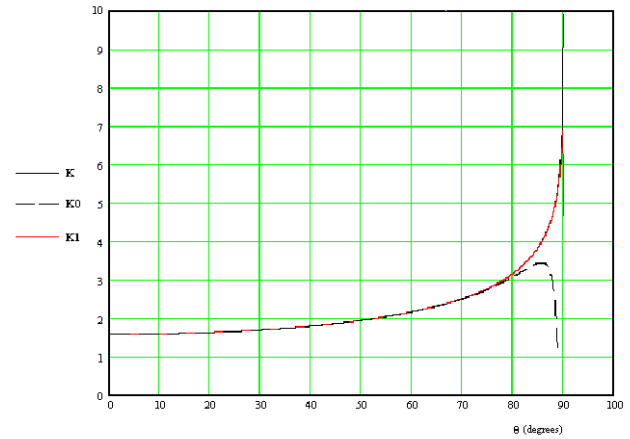


Fig. 1. Comparison of $K(k)$ with the closed analytic functions $K_0(k), K_1(k)$; also see the 2nd part of remark 1 in the appendix

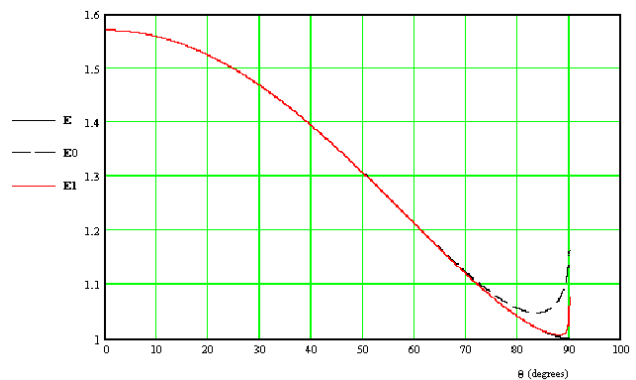


Fig. 2. Comparison of $E(k)$ with the closed analytic functions $E_0(k), E_1(k)$; also see the 2nd part of remark 1 in the appendix

VI. CONCLUSIONS

As for simplicity, the formulas in k/k' do not need mathematical tables (are purely algebraic). As for accuracy, in mathematical/technical applications, it must use the 1st set until $\theta = 70^\circ.5$ ($k = 0.94264$) only, and (for a better accuracy or a greater upper limit of the validity domain) the 2nd set, until $\theta = 88^\circ.2$ ($k = 0.99951$).

VII. NOTES; OTHER METHODS; FUTURE RESEARCH

Without the comparative tables 1 and 2, the errors table becoming so table 1, this work was published previously in a proceedings volume (scientific bulletin), in Romanian [8]. For the first English version of this work see [9], [10]. Approximations for the complete elliptic integrals based on the trapezoidal-type numerical integration formulas discussed in [11], are developed in [12], [13] (a mixed numerical-analytic method). Newer formulas (using Γ function—not an elementary, but a special one, like K & E , even if these formulas are the most accurate) are in [14], [15]; as stated in their abstracts, the works [9], [14] do not have the same goal. An original fast converging recurrent-iterative scheme to get a 3rd (and higher) set of closed analytic formulas (seemingly intricate) with desired accuracy is given in article's appendix. This article represents a fully extended version of the paper [9]. Notable special functions suitable for applying such an approximate method of calculation are: $Si(x); Ci(x); Ei(x); li(x)$.

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APPENDIX – A FAST CONVERGING
RECURRENTITERATIVE
SCHEME TO GET A 3RD (AND HIGHER) SET
OF ANALYTIC FORMULAS WITH DESIRED
ACCURACY

The formulas for transforming the modulus (Landen, [16]-[18]) are:

$$K(k) = \frac{2}{1 + \sqrt{1 - k^2}} K\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) = \frac{2}{1 + k'} K\left(\frac{1 - k'}{1 + k'}\right),$$

or: $K(\theta) = K[\tan^2(\theta/2)] / \cos^2(\theta/2)$, and, respectively:

$$E(k) = (1 + \sqrt{1 - k^2}) E\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) - \sqrt{1 - k^2} K(k) =$$

$$(1 + k') E\left[\frac{1 - k'}{1 + k'}\right] - k' K(k), \text{ with } k' = (1 - k^2)^{1/2},$$

$$\text{or: } E(\theta) = 2 \cos^2(\theta/2) \cdot E[\tan^2(\theta/2)] - \cos \theta \cdot K(\theta),$$

(passing from k to $k_1 = (1 - k)/(1 + k) \leq k$ and from θ to $\theta_1 = \sin^{-1}[\tan^2(\theta/2)] \leq \theta$, $k_1 = k(\theta_1 = \theta)$, for: $k = 0$; $1(\theta = 0; \pi/2)$), which can be transcribed in recurrent form, as follows:

$$K_2(k) = \frac{2}{1 + \sqrt{1 - k^2}} K_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) = \frac{2}{1 + k'} K_1\left(\frac{1 - k'}{1 + k'}\right),$$

or: $K_2(\theta) = K_1[\tan^2(\theta/2)] / \cos^2(\theta/2)$, and, respectively:

$$E_2(k) = (1 + \sqrt{1 - k^2}) E_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) - \sqrt{1 - k^2} K_2(k) =$$

$$= (1 + k') E_1\left(\frac{1 - k'}{1 + k'}\right) - \frac{2k'}{1 + k'} K_1\left(\frac{1 - k'}{1 + k'}\right), \text{ or: } E_2(\theta) =$$

$$2 \cos^2(\theta/2) E_1[\tan^2(\theta/2)] - [\cos \theta / \cos^2(\theta/2)] K_1[\tan^2(\theta/2)],$$

expressing the 3rd set (K_2, E_2) in function of the 2nd one (K_1, E_1), so starting a recurrent-iterative scheme; it allows writing

$$\text{for the } (n + 1)^{\text{th}} \text{ set: } K_n(k) = \frac{2}{1 + k'} K_{n-1}\left(\frac{1 - k'}{1 + k'}\right), \text{ and}$$

$$E_n(k) = (1 + k') E_{n-1}\left(\frac{1 - k'}{1 + k'}\right) - \frac{2k'}{1 + k'} K_{n-1}\left(\frac{1 - k'}{1 + k'}\right), \text{ resp.}$$

Starting from the newly found closed analytic formulas, which connect the 3rd set (K_2, E_2) with the 2nd one (K_1, E_1), by applying the new recurrent-iterative scheme previously presented, the comparative tables 1 and 2 from chapter 2 were remade, suppressing the column “ $k = \sin \theta$ ”, and inserting the new columns “ $K_2(k)$ ” and “ $E_2(k)$ ” with 4D approximate values, resp., so getting the new tables 4 and 5, given below, resp., keeping for comparison the columns “ $\theta(^{\circ})$ ”, “ $K(k)$ ”, “ $K_0(k)$ ” and “ $K_1(k)$ ” (in table 4), and “ $\theta(^{\circ})$ ”, “ $E(k)$ ”, “ $E_0(k)$ ” and “ $E_1(k)$ ” (in table 5), resp.

Table 4. Values of the functions K (part one)
(this table completes and replaces table 1)

$\theta(^{\circ})$	$K(k)$	$K_0(k)$	$K_1(k)$	$K_2(k)$
0	1.5708	1.5708	1.5708	1.5708
1	1.5709	1.5709	1.5709	1.5709
2	1.5713	1.5713	1.5713	1.5713
3	1.5719	1.5719	1.5719	1.5719
4	1.5727	1.5727	1.5727	1.5727
5	1.5738	1.5738	1.5738	1.5738
6	1.5751	1.5751	1.5751	1.5751
7	1.5767	1.5767	1.5767	1.5767
8	1.5785	1.5785	1.5785	1.5785
9	1.5805	1.5805	1.5805	1.5805
10	1.5828	1.5828	1.5828	1.5828
11	1.5854	1.5854	1.5854	1.5854
12	1.5882	1.5882	1.5882	1.5882
13	1.5913	1.5913	1.5913	1.5913
14	1.5946	1.5946	1.5946	1.5946
15	1.5981	1.5981	1.5981	1.5981
16	1.6020	1.6020	1.6020	1.6020
17	1.6061	1.6061	1.6061	1.6061
18	1.6105	1.6105	1.6105	1.6105
19	1.6151	1.6151	1.6151	1.6151
20	1.6200	1.6200	1.6200	1.6200
21	1.6252	1.6252	1.6252	1.6252
22	1.6307	1.6307	1.6307	1.6307
23	1.6365	1.6365	1.6365	1.6365
24	1.6426	1.6426	1.6426	1.6426
25	1.6490	1.6490	1.6490	1.6490
26	1.6557	1.6557	1.6557	1.6557
27	1.6627	1.6627	1.6627	1.6627
28	1.6701	1.6701	1.6701	1.6701
29	1.6777	1.6777	1.6777	1.6777
30	1.6858	1.6857	1.6858	1.6858
31	1.6941	1.6941	1.6941	1.6941
32	1.7028	1.7028	1.7028	1.7028
33	1.7119	1.7119	1.7119	1.7119
34	1.7214	1.7214	1.7214	1.7214
35	1.7312	1.7312	1.7312	1.7312
36	1.7415	1.7415	1.7415	1.7415
37	1.7522	1.7522	1.7522	1.7522
38	1.7633	1.7632	1.7633	1.7633
39	1.7748	1.7748	1.7748	1.7748
40	1.7868	1.7867	1.7868	1.7868
41	1.7992	1.7992	1.7992	1.7992
42	1.8122	1.8121	1.8122	1.8122
43	1.8256	1.8256	1.8256	1.8256
44	1.8396	1.8395	1.8396	1.8396
45	1.8541	1.8540	1.8541	1.8541
46	1.8691	1.8691	1.8691	1.8691
47	1.8848	1.8847	1.8848	1.8848
48	1.9011	1.9009	1.9011	1.9011
49	1.9180	1.9178	1.9180	1.9180
50	1.9356	1.9354	1.9356	1.9356
51	1.9539	1.9536	1.9539	1.9539
52	1.9729	1.9726	1.9729	1.9729
53	1.9927	1.9923	1.9927	1.9927
54	2.0133	2.0128	2.0133	2.0133
55	2.0347	2.0341	2.0347	2.0347
56	2.0571	2.0564	2.0571	2.0571
57	2.0804	2.0795	2.0804	2.0804
58	2.1047	2.1037	2.1047	2.1047
59	2.1300	2.1288	2.1300	2.1300
60	2.1565	2.1551	2.1565	2.1565
61	2.1842	2.1825	2.1842	2.1842
62	2.2132	2.2111	2.2132	2.2132
63	2.2435	2.2410	2.2435	2.2435
64	2.2754	2.2723	2.2754	2.2754
65	2.3088	2.3051	2.3088	2.3088
66	2.3439	2.3394	2.3439	2.3439
67	2.3809	2.3754	2.3809	2.3809
68	2.4198	2.4132	2.4198	2.4198
69	2.4610	2.4530	2.4610	2.4610
70	2.5046	2.4948	2.5045	2.5046
70.5	2.5273	2.5165	2.5273	2.5273
71	2.5507	2.5389	2.5507	2.5507
71.5	2.5749		2.5749	2.5749
72	2.5998		2.5998	2.5998
72.5	2.6256		2.6255	2.6256
73	2.6521		2.6521	2.6521
73.5	2.6796		2.6796	2.6796
74	2.7081		2.7081	2.7081
74.5	2.7375		2.7375	2.7375
75	2.7681		2.7680	2.7681
75.5	2.7998		2.7997	2.7998
76	2.8327		2.8326	2.8327
76.5	2.8669		2.8669	2.8669
77	2.9026		2.9025	2.9026
77.5	2.9397		2.9397	2.9397
78	2.9786		2.9785	2.9786
78.5	3.0192		3.0191	3.0192
79	3.0617		3.0616	3.0617
79.5	3.1064		3.1063	3.1064
80	3.1534		3.1533	3.1534
80.2	3.1729		3.1727	3.1729
80.4	3.1928		3.1927	3.1928
80.6	3.2132		3.2130	3.2132
80.8	3.2340		3.2338	3.2340
81	3.2553		3.2551	3.2553

Table 4. Values of the functions K (part two)

81.2	3.2771	3.2769	3.2771
81.4	3.2995	3.2992	3.2995
81.6	3.3223	3.3221	3.3223
81.8	3.3458	3.3455	3.3458
82	3.3699	3.3696	3.3699
82.2	3.3946	3.3942	3.3946
82.4	3.4199	3.4196	3.4199
82.6	3.4460	3.4456	3.4460
82.8	3.4728	3.4724	3.4728
83	3.5004	3.4999	3.5004
83.2	3.5288	3.5283	3.5288
83.4	3.5581	3.5575	3.5581
83.6	3.5884	3.5877	3.5884
83.8	3.6196	3.6188	3.6196
84	3.6519	3.6510	3.6519
84.2	3.6852	3.6843	3.6852
84.4	3.7198	3.7187	3.7198
84.6	3.7557	3.7545	3.7557
84.8	3.7930	3.7916	3.7930
85	3.8317	3.8302	3.8317
85.2	3.8721	3.8704	3.8721
85.4	3.9142	3.9122	3.9142
85.6	3.9583	3.9560	3.9583
85.8	4.0044	4.0018	4.0044
86	4.0528	4.0498	4.0528
86.2	4.1037	4.1003	4.1037
86.4	4.1574	4.1535	4.1574
86.6	4.2142	4.2097	4.2142
86.8	4.2744	4.2692	4.2744
87	4.3387	4.3325	4.3387
87.2	4.4073	4.4001	4.4073
87.4	4.4811	4.4726	4.4811
87.6	4.5609	4.5507	4.5609
87.8	4.6477	4.6354	4.6477
88	4.7427	4.7277	4.7427
88.2	4.8478	4.8293	4.8478
88.4	4.9654		4.9654
88.6	5.0988		5.0987
88.8	5.2527		5.2527
89	5.4349		5.4349
89.1	5.5402		5.5402
89.2	5.6579		5.6579
89.3	5.7914		5.7913
89.4	5.9455		5.9454
89.5	6.1278		6.1276
89.6	6.3509		6.3506
89.7	6.6385		6.6380
89.8	7.0440		7.0428
89.9	7.7371		7.7336
90	∞	- ∞	- ∞

The values string in the last column is given by:

$$K_2(k) = \frac{2}{1 + \sqrt{1 - k^2}} K_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) = \frac{2}{1 + k'} K_1\left(\frac{1 - k'}{1 + k'}\right),$$

$$\text{with: } K_1(k_1) = \frac{\pi\sqrt{2}}{\sqrt{(1+k'_1)\sqrt{k'_1}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1 + \sqrt{k'_1}}{\sqrt[4]{(1+k'_1)\sqrt{k'_1}}}\right) = \frac{\pi\sqrt{2}}{\sqrt{\left(1 + \frac{2\sqrt{k'}}{1+k'}\right)\sqrt{1+k'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1 + \frac{\sqrt{2\sqrt{k'}}}{\sqrt{1+k'}}}{\sqrt[4]{\left(1 + \frac{2\sqrt{k'}}{1+k'}\right)\sqrt{1+k'}}}\right),$$

and finally the algebraic formula: $K_2(k) = 2K_1(k_1)/(1 + k')$.

Table 5. Values of the functions E (part one)
(this table completes and replaces table 2)

$\theta(^{\circ})$	$E(k)$	$E_0(k)$	$E_1(k)$	$E_2(k)$
0	1.5708	1.5708	1.5708	1.5708
1	1.5707	1.5707	1.5707	1.5707
2	1.5703	1.5703	1.5703	1.5703
3	1.5697	1.5697	1.5697	1.5697
4	1.5689	1.5689	1.5689	1.5689
5	1.5678	1.5678	1.5678	1.5678
6	1.5665	1.5665	1.5665	1.5665
7	1.5649	1.5649	1.5649	1.5649
8	1.5632	1.5632	1.5632	1.5632
9	1.5611	1.5611	1.5611	1.5611
10	1.5589	1.5589	1.5589	1.5589
11	1.5564	1.5564	1.5564	1.5564
12	1.5537	1.5537	1.5537	1.5537
13	1.5507	1.5507	1.5507	1.5507
14	1.5476	1.5476	1.5476	1.5476
15	1.5442	1.5442	1.5442	1.5442
16	1.5405	1.5405	1.5405	1.5405
17	1.5367	1.5367	1.5367	1.5367
18	1.5326	1.5326	1.5326	1.5326
19	1.5283	1.5283	1.5283	1.5283
20	1.5238	1.5238	1.5238	1.5238
21	1.5191	1.5191	1.5191	1.5191
22	1.5141	1.5141	1.5141	1.5141
23	1.5090	1.5090	1.5090	1.5090
24	1.5037	1.5037	1.5037	1.5037
25	1.4981	1.4981	1.4981	1.4981
26	1.4924	1.4924	1.4924	1.4924
27	1.4864	1.4864	1.4864	1.4864
28	1.4803	1.4803	1.4803	1.4803
29	1.4740	1.4740	1.4740	1.4740
30	1.4675	1.4675	1.4675	1.4675
31	1.4608	1.4608	1.4608	1.4608
32	1.4539	1.4539	1.4539	1.4539
33	1.4469	1.4469	1.4469	1.4469
34	1.4397	1.4397	1.4397	1.4397
35	1.4323	1.4323	1.4323	1.4323
36	1.4248	1.4248	1.4248	1.4248
37	1.4171	1.4171	1.4171	1.4171
38	1.4092	1.4093	1.4092	1.4092
39	1.4013	1.4013	1.4013	1.4013
40	1.3931	1.3932	1.3931	1.3931

Table 5. Values of the functions E (part two)

					81	1.0338		1.0339	1.0338
41	1.3849	1.3849	1.3849	1.3849	81.2	1.0326		1.0327	1.0326
42	1.3765	1.3765	1.3765	1.3765	81.4	1.0314		1.0315	1.0314
43	1.3680	1.3680	1.3680	1.3680	81.6	1.0302		1.0303	1.0302
44	1.3594	1.3594	1.3594	1.3594	81.8	1.0290		1.0292	1.0290
45	1.3506	1.3507	1.3506	1.3506	82	1.0278		1.0280	1.0278
46	1.3418	1.3419	1.3418	1.3418	82.2	1.0267		1.0269	1.0267
47	1.3329	1.3330	1.3329	1.3329	82.4	1.0256		1.0258	1.0256
48	1.3238	1.3239	1.3238	1.3238	82.6	1.0245		1.0247	1.0245
49	1.3147	1.3148	1.3147	1.3147	82.8	1.0234		1.0236	1.0234
50	1.3055	1.3057	1.3055	1.3055	83	1.0223		1.0226	1.0223
51	1.2963	1.2964	1.2963	1.2963	83.2	1.0213		1.0215	1.0213
52	1.2870	1.2872	1.2870	1.2870	83.4	1.0202		1.0205	1.0202
53	1.2776	1.2778	1.2776	1.2776	83.6	1.0192	false min.	1.0196	1.0192
54	1.2681	1.2684	1.2681	1.2681	83.8	1.0182		1.0186	1.0182
55	1.2587	1.2590	1.2587	1.2587	84	1.0172		1.0176	1.0172
56	1.2492	1.2496	1.2492	1.2492	84.2	1.0163		1.0167	1.0163
57	1.2397	1.2401	1.2397	1.2397	84.4	1.0153		1.0158	1.0153
58	1.2301	1.2307	1.2301	1.2301	84.6	1.0144		1.0150	1.0144
59	1.2206	1.2212	1.2206	1.2206	84.8	1.0135		1.0141	1.0135
60	1.2111	1.2118	1.2111	1.2111	85	1.0127		1.0133	1.0127
61	1.2015	1.2024	1.2015	1.2015	85.2	1.0118		1.0125	1.0118
62	1.1920	1.1930	1.1920	1.1920	85.4	1.0110		1.0118	1.0110
63	1.1826	1.1838	1.1826	1.1826	85.6	1.0102		1.0110	1.0102
64	1.1732	1.1745	1.1732	1.1732	85.8	1.0094		1.0103	1.0094
65	1.1638	1.1654	1.1638	1.1638	86	1.0086		1.0097	1.0086
66	1.1545	1.1564	1.1545	1.1545	86.2	1.0079		1.0091	1.0079
67	1.1453	1.1475	1.1453	1.1453	86.4	1.0072		1.0085	1.0072
68	1.1362	1.1387	1.1362	1.1362	86.6	1.0065		1.0080	1.0065
69	1.1272	1.1301	1.1273	1.1272	86.8	1.0059		1.0075	1.0059
70	1.1184	1.1217	1.1184	1.1184	87	1.0053		1.0071	1.0053
70.5	1.1140	1.1176	1.1140	1.1140	87.2	1.0047		1.0067	1.0047
71	1.1096	1.1135	1.1096	1.1096	87.4	1.0041		1.0064	1.0041
71.5	1.1053		1.1053	1.1053	87.6	1.0036		1.0062	1.0036
72	1.1011		1.1011	1.1011	87.8	1.0031		1.0060	1.0031
72.5	1.0968		1.0968	1.0968	88	1.0026		1.0060	1.0026
73	1.0927		1.0927	1.0927	88.2	1.0021		1.0061	1.0021
73.5	1.0885		1.0885	1.0885	88.4	1.0017			1.0017
74	1.0844		1.0844	1.0844	88.6	1.0014			1.0014
74.5	1.0804		1.0804	1.0804	88.8	1.0010			1.0011
75	1.0764		1.0764	1.0764	89	1.0008			1.0008
75.5	1.0725		1.0725	1.0725	89.1	1.0006			1.0006
76	1.0686		1.0686	1.0686	89.2	1.0005			1.0005
76.5	1.0648		1.0648	1.0648	89.3	1.0004			1.0004
77	1.0611		1.0611	1.0611	89.4	1.0003			1.0003
77.5	1.0574		1.0574	1.0574	89.5	1.0002			1.0003
78	1.0538		1.0538	1.0538	89.6	1.0001			1.0002
78.5	1.0502		1.0503	1.0502	89.7	1.0001			1.0002
79	1.0468		1.0468	1.0468	89.8	1.0000			1.0003
79.5	1.0434		1.0435	1.0434	89.9	1.0000			1.0007
80	1.0401		1.0402	1.0401	90	1.0000	1.1781	1.1781	1.1781
80.2	1.0388		1.0389	1.0388	The values string in the last column is given by:				
80.4	1.0375		1.0376	1.0375	$E_2(k) = (1 + \sqrt{1 - k^2}) E_1 \left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}} \right) - \sqrt{1 - k^2} K_2(k) =$				
80.6	1.0363		1.0364						
80.8	1.0350		1.0351						
			1.0350						

$$\begin{aligned}
 &= (1+k')E_1\left(\frac{1-k'}{1+k'}\right) - k'K_2(k) = \\
 &= (1+k')E_1\left(\frac{1-k'}{1+k'}\right) - \frac{2k'}{1+k'}K_1\left(\frac{1-k'}{1+k'}\right), \text{ with:} \\
 \frac{1-k'}{1+k'} &= k_1 \text{ (descending Landen transformation), getting} \\
 E_1(k_1) &= \frac{\pi}{4} \left[\frac{3}{2} (1 + \sqrt{k_1'})^2 - \sqrt{2} \sqrt{1+k_1'} \sqrt[4]{k_1'} \right] - k_1' \cdot K_1(k_1), \\
 \text{and: } K_1(k_1) &= \frac{\pi\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}} \right),
 \end{aligned}$$

previously given, thus getting:

$$\begin{aligned}
 E_1(k_1) &= \frac{\pi}{4} \left[\frac{3}{2} (1 + \sqrt{k_1'})^2 - \sqrt{2} \sqrt{(1+k_1')\sqrt{k_1'}} \right] - \\
 &\quad - \frac{\pi k_1' \sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}} \right) = \\
 &= \frac{\pi}{4} \left[\frac{3}{2} (1 + \sqrt{k_1'})^2 - \sqrt{2} (1+k_1') \sqrt{k_1'} - \right. \\
 &\quad \left. - \frac{k_1' \sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}} \left(4 - \frac{\sqrt[4]{2} (1 + \sqrt{k_1'})}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}} \right) \right].
 \end{aligned}$$

Expressing $k_1'(k)$: $k_1' = (1 - k_1^2)^{1/2} = 2(k)^{1/2}/(1+k)$, (ascending Landen transformation), and replacing it:

$$\begin{aligned}
 E_1(k_1) &= \frac{\pi}{4} \left[\frac{3}{2} \left(1 + \frac{\sqrt{2\sqrt[4]{k'}}}}{\sqrt{1+k'}} \right)^2 - \sqrt{2} \left(1 + \frac{2\sqrt{k'}}{1+k'} \right) \frac{\sqrt{2\sqrt[4]{k'}}}}{\sqrt{1+k'}} \right. \\
 &\quad \left. - \frac{\sqrt{2} \cdot \frac{2\sqrt{k'}}{1+k'}}{\sqrt{\left(1 + \frac{2\sqrt{k'}}{1+k'} \right) \frac{\sqrt{2\sqrt[4]{k'}}}}{\sqrt{1+k'}}}} \left(4 - \frac{\sqrt[4]{2} \left(1 + \frac{\sqrt{2\sqrt[4]{k'}}}}{\sqrt{1+k'}} \right)}{\sqrt[4]{\left(1 + \frac{2\sqrt{k'}}{1+k'} \right) \frac{\sqrt{2\sqrt[4]{k'}}}}{\sqrt{1+k'}}}} \right) \right],
 \end{aligned}$$

and finally: $E_2(k) = (1+k')E_1(k_1) - k'K_2(k)$, where $K_2(k)$ was given just before table 5, so getting another purely algebraic formula (the most accurate, just seemingly intricate), the 3rd set of closed analytic formulas being given by the recurrences: $K_2(k) = 2K_1(k_1)/(1+k)$; $E_2(k) = (1+k')E_1(k_1) - k'K_2(k)$. Noting: $k_1' = x$ and $[(1+x) \cdot x^{1/2}]^{1/2} = y$, one can write:

$$\begin{aligned}
 K_2(k) &= \pi(2/k')^{1/2} \cdot (x/y) \left[1 - (2^{1/4}/4)(1+x^{1/2})/y^{1/2} \right]; \\
 E_2(k) &= \pi(k')^{1/2}/(2x) \cdot \left\{ (3/2)(1+x^{1/2})^2 - 2^{1/2}y - \right. \\
 &\quad \left. - 2^{1/2}(x/y) \left[4 - 2^{1/4}(1+x^{1/2})/y^{1/2} \right] \right\} - k'K_2(k), \text{ resp.,}
 \end{aligned}$$

much simpler than previous ones (for calculation only). The validity of all approximate sets is limited to $k \in [0, k_{\text{extr}}]$; $k_{\text{extr}} \leq 1$, “extr” \equiv extremum (max. for K , and min. for E ; $k_{\text{max}} \neq k_{\text{min}}$) (see figs. 1 & 2 – the dashed black lines, and the solid red ones, resp.). The higher the “n” index (of the (K_{n-1}, E_{n-1}) approximation set) value is, the better this approximation is and the closer to the right domain’s end ($k = 1$) the extremum is located. We will cancel the recurrent-iterative scheme (stopping it to a specific “n” index value) when the maximum relative

error (over the whole valid domain of variation $k \in [0, k_{\text{extr}}]$) becomes lesser than the desired (required) accuracy. The first important application of the results obtained in chapter 4 consists in determining the locations of the extrema values k_{extr} (k_{max} for $K_{n-1}(k)$ and k_{min} for $E_{n-1}(k)$), corresponding to the annulment of their first derivatives with respect to k , using the relations: $K'_{n-1}(k) = dK_{n-1}(k)/dk = 0$; $E'_{n-1}(k) = dE_{n-1}(k)/dk = 0$, and adding the recurrent definitions for $K_{n-1}(k)$ and $E_{n-1}(k)$. The 1st ODE above gives the value k_{max} and the 2nd one gives the value k_{min} . Each of these ODEs has really two solutions. Besides the searched for one, both ODEs admit the solution $k = 0$, corresponding to a minimum for $K_{n-1}(k)$ and to a maximum for $E_{n-1}(k)$, both with the value $\pi/2$ (for both approximate and exact functions: $K_{n-1}(0) = E_{n-1}(0) = K(0) = E(0) = \pi/2$, with: $K'_{n-1}(0) = E'_{n-1}(0) = K'(0) = E'(0) = 0$, but with: $K''_{n-1}(0) > 0$ and $K''(0) > 0$ – a minimum, while: $E''_{n-1}(0) < 0$ and $E''(0) < 0$ – a maximum).

Thus one knows now the values k_{max} and k_{min} (the right ends of the validity domains of the approximate functions). In order to evaluate the accuracy of the 3rd set (K_2, E_2), similarly as for the previous two sets, (K_0, E_0) and (K_1, E_1), we will define the following relative error functions: $\varepsilon_{K_2}(k) = K_2(k)/K(k) - 1$, and: $\varepsilon_{E_2}(k) = E_2(k)/E(k) - 1$, for the approximate formulas of 1st & 2nd kind integrals. Their values are given in table 6, expressed in thousandths (%). These errors were calculated for the 3rd set (K_2, E_2) only, with an increment of $0^\circ.2$ in the field $\theta \in [84^\circ, 89^\circ]$ of the domain, and of $0^\circ.1$ beyond 89° . To get table 6, in table 3 were suppressed the columns $\varepsilon_{K_0}(\%)$, $\varepsilon_{E_0}(\%)$ (the most inaccurate) and were inserted the columns $\varepsilon_{K_2}(\%)$, $\varepsilon_{E_2}(\%)$, keeping for comparison the columns “ $\theta(^\circ)$ ”, “ $k = \sin \theta$ ”, “ $\varepsilon_{K_1}(\%)$ ” and “ $\varepsilon_{E_1}(\%)$ ” (from table 3), only.

Table 6. Relative errors ε distribution (this table completes and replaces table 3)

$\theta(^\circ)$	$k = \sin \theta$	$\varepsilon_{K_1}(\%)$	$\varepsilon_{K_2}(\%)$	$\varepsilon_{E_1}(\%)$	$\varepsilon_{E_2}(\%)$
84.8	0.99588	-0.369	0	+0.607	0
85	0.99619	-0.396	0	+0.592	0
85.2	0.99649	-0.451	0	+0.705	0
85.4	0.99678	-0.500	0	+0.748	0
85.6	0.99705	-0.582	0	+0.823	0
85.8	0.99731	-0.652	0	+0.932	0
86	0.99756	-0.737	0	+1.076	0
86.2	0.99780	-0.832	0	+1.160	0
86.4	0.99803	-0.945	0	+1.284	0
86.6	0.99824	-1.077	0	+1.453	0
86.8	0.99844	-1.214	0	+1.571	0
87	0.99863	-1.421	0	+1.743	0
87.2	0.99881	-1.626	0	+1.976	0
87.4	0.99897	-1.894	0	+2.275	0
87.6	0.99912	-2.234	0	+2.553	0
87.8	0.99926	-2.655	0	+2.922	0
88	0.99939	-3.156	0	+3.397	0
88.2	0.99951	-3.808	0	+4.004	0

88.4	0.99961	-	0	-	0
88.6	0.99970	-	0	-	0
88.8	0.99978	-	0	-	0
89	0.99985	-	0	-	0
89.1	0.99988	-	0	-	0
89.2	0.99990	-	0	-	0
89.3	0.99993	-	0	-	0
89.4	0.99995	-	0	-	0
89.5	0.99996	-	-	-	-
89.6	0.99998	-	-	-	-
89.7	0.99999	-	-	-	-
89.8	0.99999	-	-	-	-
89.9	1.00000	-	-	-	-
90	1.00000	-2000	-2000	178.097	178.097

The errors strings are stopped if their modulus is $\geq 4\%$. From the tables 3 and 6 one can see that, for any n^{th} set of approximation and at any k value, $\varepsilon_K < 0$ ($K_n < K$) and $\varepsilon_E > 0$ ($E_n > E$), i.e. K is approximated by lack, while E – by excess. Similarly to the 3rd set $[K_2(k), E_2(k)]$, expressed in algebraic functions, one can build the 3rd set $[K_2(\theta), E_2(\theta)]$, expressed in trigonometric functions, replacing k' in $[K_2(k), E_2(k)]$ set by $\cos \theta$ and applying usual trigonometric identities. The comparative series representations and the graphic comparison are superfluous, due to the great accuracy of the approximate values given by the 3rd set (practically identical to the exact ones, which could be already noticed from the analysis of the 2nd set, this showing the fast converging character of this recurrent-iterative scheme). Except for the right domain's end ($k = 1$), the 3rd set of approximation (K_2, E_2), even more accurate than the 2nd one (K_1, E_1), may be considered and successfully used instead of the exact values of $K(k)$ and $E(k)$ from mathematical tables. A false minimum takes place for all $E_n(k)$: for $E_2(k)$, at $\theta = 89^\circ.7$ ($k = 0.99999$); for $E_1(k)$, at $\theta = 88^\circ$ ($k = 0.99939$), and for $E_0(k)$, at $\theta = 83^\circ.62$ ($k = 0.99381$). The graphs of all $E_n(k)$ pass through the point $(1, 3\pi/8 = 1.178097)$; for k tending to unity, the graphs of all $K_n(k)$ go toward $(-\infty)$; the higher n^{th} sets ($n \geq 4$) give a much better accuracy). Unlike the mathematical tables (and in addition to them), all approximation sets (the 1st, 2nd, 3rd and the higher n^{th} ($n \geq 4$) ones) allow performing the *analytic study of variation* of the functions in which $K(k)$ and / or $E(k)$ appear /s, using *the derivatives of the 1st and 2nd order* (with respect to k). *Remarks:* 1. As a first step in applying the new recurrent-iterative scheme, just the obtaining of the 2nd set (K_1, E_1) as a function of the 1st one (K_0, E_0) (in ch. 2) may be considered, i.e. *this scheme starts really at the 2nd set*. It is to be highlighted the used method is a *purely analytic* one (neither numerical methods nor sophisticated software, at most using MatLab's (software package for engineers) "Symbolic Math" toolbox, for analytically solving the more intricate algebraic equations encountered). Its *simplicity, accuracy and fast convergence*, as well as its *limitations* depend *exclusively* on the *correct choice of its starting point* (approximation set) (K_0, E_0). It must be quite precise, and especially, *as simple as possible*.

The starting approximate formula-definition giving $E_0(k)$ was suggested to the author by an old approximate formula (Peano, [19], [20]) for the perimeter L of an ellipse of semi-axes a and b ($b \leq a$): $L \approx \pi[1.5(a + b) - (ab)^{1/2}]$ – a good (& simple) approximation with the best accuracy for $b = a$ (circle): $L = 2\pi a$, and the worst one for $b = 0$ (plane plate): $L = 1.5\pi a$, instead of $L = 4a$ (or optimized Peano's law: $L_1 \approx \pi[1.32(a + b) - 0.64(ab)^{1/2}]$, with the smallest overall error [21](about 7 times smaller than that of the original law); for $b = a$: $L_1 = L = 2\pi a$, and for $b = 0$: $L_1 = 1.32\pi a$, much closer to the exact value $L = 4a$). For its behaviour at low b/a ratios, this formula is not found on the list of the very accurate (but not simple) approximations [21] (Padé, Jacobsen, Ramanujan (2 expressions), Rackauckas), all expressed in terms of the particular ratio $h = [(a - b)/(a + b)]^2$. Thus a reliable approximate (by excess) formula-definition was obtained (see chapter 2) for the Legendre complete elliptic integral of the 2nd kind (in the 1st set of approximation): $E_0(k) = (\pi/4)[1.5(1 + k') - (k')^{0.5}]$, with $k' = (1 - k^2)^{0.5}$. It can be seen that the error committed if in the expansion in series of powers (of k) we stopped at the term of rank 5 (see chapter 4), is $(3/16384)k^8$ only, i.e. small enough. As for the pair approximate formula-definition giving $K_0(k)$, this was obtained using the previous one for $E_0(k)$ and applying the definition of the first derivative of $E(k)$ with respect to k : $dE(k)/dk = [E(k) - K(k)]/k$ (see chapter 4), thus getting: $K(k) = E(k) - k[dE(k)/dk]$; replacing $K(k)$ and $E(k)$ by their 1st approximations: $K_0(k)$ and the previously given $E_0(k)$, one gets: $K_0(k) = (\pi/8)[3/2(1 + 1/k') - (k')^{0.5}(1 + 1/(k')^2)]$, of a lesser accuracy (esp. for $\theta > \pi/3$) than $E_0(k)$. To improve this, one uses a descending Landen transformation: $K(k) = (1 + k_1)K(k_1)$ with $k_1 = (1 - k')/(1 + k') \leq k$, and replacing in $K(k)$, one gets: $K_0(k) = \pi[1/(k')^{0.5} - (1/2^{1.5})(1 + k')^{0.5}/(k')^{0.75}] \geq K_0(k)$ (see ch. 2), of an accuracy (in modulus) much closer to that of its pair $E_0(k)$. Being practically generated by the same mathematical source, $K_0(k)$ and $E_0(k)$ vary (ordinates, slopes, asymptote, extrema, concavities, convexities, inflections) in perfectly correlated way. So, at the value k_{extr} corresponding to a false minimum for $E_0(k)$, $K_0(k)$ must equate $E_0(k)$, to satisfy the annulment of $dE_0(k)/dk$. To prepare this, $K_0(k)$ must stop its vertiginous ascension to ∞ , making a false inflection, followed by a false max. at $k_{\text{extr}} < k_{\text{extr}}$ and a vertiginous ($k = 1 -$ vertical asymptote) fall toward $(-\infty)$; so $K_0 = E_0$ at $k = 0$ and $k = k_{\text{extr}}$. But the new more accurate K_0 is not generated by the same mathematical source as E_0 . To minimise the unwished events, limiting them to a very thin region in the neighbourhood of the right domain's end, one applies the descending Landen transformation, passing from k to $k_1 \leq k$, where all goes well, maintaining all advantages of the asymptotic behaviour of the new approximate functions (K_n, E_n), i.e. applying a higher n^{th} ($n \geq 2$) set of approximation (repeating this scheme until the desired accuracy for (K_n, E_n) is obtained; fortunately, this scheme is fast converging); though it keeps the limitation at $k = 1$, Peano's optimized law accelerates the scheme. 2. Besides the formulas for transforming the modulus using the descending Landen transformation, there are formulas using the ascending Landen transformation (not of interest here).

Appendix' conclusions

Some authors (e.g.: Bagis [14], [15]) choose to start from more precise formulas for the perimeter of an ellipse (similar to Ramanujan's "type π formulas" (1914) – see [22]): $L_I = \pi \{3(a+b) - [(a+3b)(3a+b)]^{1/2}\} = \pi \{3(a+b) - [10ab + 3(a^2 + b^2)]^{1/2}\}$ – Ramanujan 1st approximation; $L_{II} = \pi(a+b)\{1 + 3h/[10 + (4-3h)^{1/2}]\}$; $h = [(a-b)/(a+b)]^2$ – the more famous Ramanujan 2nd approximation; the errors in these empirical relations, are of order h^3 and h^5 (both being very accurate, but not as simple as possible), in order to obtain approximate formulas as accurate as possible for Legendre's complete elliptic integrals.

We cite from [21]: "What makes Ramanujan's first formula interesting to this Author is the fact that, like the first form of Peano's approximation, it can be interpreted as a combination of the arithmetic mean with another one, denoted as $R(a, b, w)$ and defined by: $R(a, b, w) = [(a+wb)(b+wa)]^{1/2}/(1+w)$. In Ramanujan's formula we have $w=3$ and the two means are combined linearly with the relative weights $+3$ and -2 , resp." Noteworthy are the fast converging power (of h) series [23], [24]. This appendix demonstrates that even choosing as a starting point a "not so precise" (with big problems at the right domain's end $k=1$), but especially simple formula (like Peano's, or better, optimized Peano's one), and applying the newly found original fast converging recurrent-iterative scheme (also including Landen's descending transformation, to solve the unwished behaviour of $E_n(k)$ appeared in the neighbourhood of the value $k=1$ of the modulus (the right domain's end), due to any of both Peano's approximate laws), this being a major method's limitation (see the 2nd part of remark 1), similar results (from the viewpoint of their accuracy) for the values of Legendre's complete elliptic integrals $K(k)$ and $E(k)$ (with very small values of the relative errors ε_K and ε_E – practically zero) can be obtained.

As regards the relations describing the recurrence (for the $(n+1)$ th set of approximation $[K_n(k), E_n(k)]$), they are: $K_n(k) = [2/(1+k')]K_{n-1}(k_1)$, and: $E_n(k) = (1+k')E_{n-1}(k_1) - [2k'/(1+k')]K_{n-1}(k_1)$, resp., where: $k' = (1-k^2)^{1/2}$ is the complementary modulus, and: $k_1 = (1-k)/(1+k) \leq k$, this representing just the source of the descending Landen transformation; they express the values of the $(n+1)$ th set $[K_n(k), E_n(k)]$ in function of those of the n th one $[K_{n-1}(k_1), E_{n-1}(k_1)]$. The iterative scheme can continue until the desired (required) accuracy for the approximate set (K_n, E_n) at a considered value of modulus $k = \sin \theta$ is obtained. As a rule, in the practical applications, the 3rd set of approximation (K_2, E_2) is sufficiently accurate. It can be used until $\theta = 89^\circ.7$ ($k = 0.99999$) – also see tables 4 – 6. Though it keeps the limitation at $k=1$, Peano's optimized law is better to use; perhaps an example of calculation would have been useful, but we took as "overwhelming" its quality stated in [21]. Without these "appendix' conclusions", this work was published previously in a unitary form (main article + appendix), in English, as a scientific paper [25].

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